



# On a three step model of anaerobic digestion including the hydrolysis of particulate matter

R. Fekih-Salem, Nahla Abdellatif, T. Sari, Jérôme Harmand

## ► To cite this version:

R. Fekih-Salem, Nahla Abdellatif, T. Sari, Jérôme Harmand. On a three step model of anaerobic digestion including the hydrolysis of particulate matter. MATHMOD 2012 - 7th Vienna International Conference on Mathematical Modelling, Jan 2012, Vienne, Austria. 6 p. hal-00777559

**HAL Id: hal-00777559**

**<https://hal.science/hal-00777559>**

Submitted on 17 Jan 2013

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# On a Three Step Model of Anaerobic Digestion Including the Hydrolysis of Particulate Matter

R. Fekih-Salem<sup>\*,\*\*</sup> N. Abdellatif<sup>\*\*,\*\*\*</sup> T. Sari<sup>\*,\*\*\*\*</sup>  
J. Harmand<sup>\*,†</sup>

<sup>\*</sup> *EPI Modemic Inra-Inria, UMR Mistea, SupAgro, 34060 Montpellier, France.*

<sup>\*\*</sup> *Université de Tunis El Manar, Ecole Nationale d'Ingénieurs de Tunis, LAMSIN, 1002 Tunis Le Belvédère, Tunisie (e-mail: fekih@supagro.inra.fr).*

<sup>\*\*\*</sup> *Ecole Nationale des Sciences de l'Informatique, Campus Universitaire de Manouba, 2010 Manouba, Tunisie (e-mail: nahla.abdellatif@ensi.rnu.tn).*

<sup>\*\*\*\*</sup> *Irstea, UMR Itap, 34196 Montpellier Cedex, France (e-mail: tewfik.sari@irstea.fr).*

<sup>†</sup> *Inra, Laboratoire de Biotechnologie de l'Environnement, 11100 Narbonne, France (e-mail: harmand@supagro.inra.fr).*

---

**Abstract:** In this work, we focus on the mathematical analysis of the model of chemostat with enzymatic degradation of a substrate (organic matter) that can partly be under a solid form Simeonov and Stoyanov (2003). The study of this 3-step model is derived from a smaller order sub-model since some variables can be decoupled from the others. We study the existence and the stability of equilibrium points of the sub-model considering both Monod or Haldane growth rates and distinct dilution rates. In the classical chemostat model with monotonic kinetics, it is well known that only one equilibrium point attracts all solutions and that bistability never occurs Smith and Waltman (1995). In the present study, although (i) only monotonic growth rates are considered and (ii) the concentrations of input substrate concentration is less than the break-even concentration, it is shown that the considered sub-model may exhibit bistability. Hence, the importance of hydrolysis in the appearance of positive equilibrium points and the bistability is pointed out. If a non monotonic growth rate is considered, depending on the input substrate concentration, it is shown that at most four positive equilibrium points exist. Furthermore, for any positive initial condition, the solution converges towards one of the positive equilibrium points for which the washout is unstable. Finally, we study the case where the growth rate is density-dependent, such as the Contois kinetics, which may be of interest if we consider that we work in a non homogeneous environment Lobry and Harmand (2006). Depending on the input substrate concentration, we show that the system can exhibit either a bistability or the global stability of the positive equilibrium point or of the washout.

---

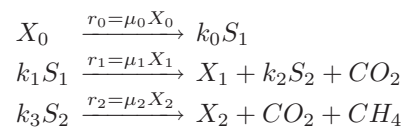
**Keywords:** Enzymatic degradation, chemostat, models, growth rate, equilibrium, bistability.

---

## 1. INTRODUCTION

Anaerobic digestion is a biological process in which organic matter is transformed into methane and carbon dioxide (biogas) by microorganisms in the absence of oxygen. The search for models simple enough to be used for control design is of prior importance today to optimize fermentation processes and solve important problems such as the development of renewable energy from waste. Within the studies of microbiology, biochemistry and technology, the anaerobic digestion is generally considered as a three step process: hydrolysis and liquefaction of the large, insoluble organic molecules by extracellular enzymes, acid production by an acidogenic microbial consortium and a methane production stage realized by a methanogenic

ecosystem. Several mathematical models describing these phenomena have been proposed in the literature. However, they are usually too complex to be used for control synthesis Simeonov and Stoyanov (2003); B. Benyahia and Harmand (2010a,b); Bastin and Dochain (1991). The chemical reactions of anaerobic digestion which converts the substrate into biomass is :



where  $r_i = \mu_i X_i, i = 0 \dots 2$ , denotes the reaction rate, respectively,  $\mu_0$  is the specific growth rate of  $X_1$  on  $X_0$  and  $\mu_i$  is the specific growth rate of  $X_i$  on  $S_i$  for  $i = 1, 2$ .

$k_i, i = 0 \dots 3$ , denote the pseudo-stoichiometric coefficients associated to the chemical reactions,

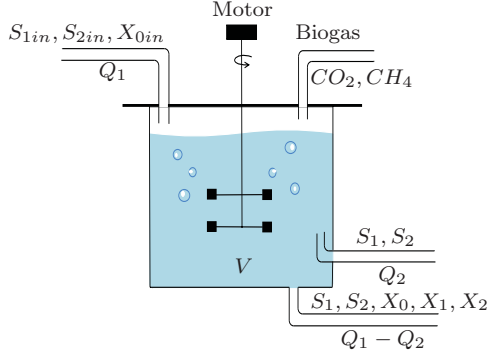


Fig. 1. Chemostat.

We consider a continuous culture, i.e the input flow rate  $Q_1$  is equal to the output flow rate. For low concentrations of substrate, the biomass residence time is greater than the substrate one, then the output flow rate of biomass and substrate in the form macromolecules is  $Q_1 - Q_2$ . The three step model Simeonov and Stoyanov (2003) is :

$$\begin{cases} \dot{X}_0 = DX_{0in} - \alpha DX_0 - \mu_0(X_0)X_1, \\ \dot{S}_1 = D(S_{1in} - S_1) + k_0\mu_0(X_0)X_1 - k_1\mu_1(S_1)X_1, \\ \dot{X}_1 = (\mu_1(S_1) - \alpha D)X_1, \\ \dot{S}_2 = D(S_{2in} - S_2) + k_2\mu_1(S_1)X_1 - k_3\mu_2(S_2)X_2, \\ \dot{X}_2 = (\mu_2(S_2) - \alpha D)X_2, \end{cases} \quad (1)$$

where

$$D = \frac{Q_1}{V} \quad \text{and} \quad \frac{Q_1 - Q_2}{V} = \alpha D.$$

$D$  denotes the dilution rate of the chemostat and  $\alpha \in [0, 1]$  represents the fraction of the biomass leaving the reactor.  $V$  denotes the volume of the bioreactor,  $X_0(t)$  the concentration of the substrate in the form macromolecules at time  $t$ , with  $X_{0in}$  the concentration of the input nutriment.  $S_j(t)$  denote the concentration of the substrates in the effluent,  $j = 1, 2$ , at time  $t$ ; with  $S_{jin}$  the input substrate concentrations  $j$ .  $X_i(t)$  denote the concentration of the  $i$ th population of microorganisms,  $i = 1, 2$ , at time  $t$ .

According to the principle of conservation of matter within the reaction scheme we have

$$\int_{t_1}^{t_2} \mu_0(X_0)X_1 V d\tau \geq \int_{t_1}^{t_2} k_0\mu_0(X_0)X_1 V d\tau \quad \text{i.e.} \quad 1 \geq k_0,$$

which means that, the quantity of  $X_0$  degraded is greater than or equal to the quantity of  $S_1$  produced. Similarly, we have

$$k_1 \geq 1 + k_2 \quad \text{and} \quad k_3 \geq 1$$

which means that, the quantity of  $S_1$  degraded is greater than or equal to the quantity of  $X_1$  and  $S_2$  produced. The quantity of  $S_2$  degraded is greater than or equal to the quantity of  $X_2$  produced.

In the following, we focus on the study of the sub-model given by the first three equations of system (1), the last two equations being decoupled since the first three equations are independent of variables  $X_2$  and  $S_2$ . Thus, we study the existence and stability of equilibrium points of the following sub-model :

$$\begin{cases} \dot{X}_0 = D(X_{0in} - \alpha X_0) - \mu_0(X_0)X_1, \\ \dot{S}_1 = D(S_{1in} - S_1) + k_0\mu_0(X_0)X_1 - k_1\mu_1(S_1)X_1, \\ \dot{X}_1 = (\mu_1(S_1) - \alpha D)X_1. \end{cases} \quad (2)$$

First, we establish the following result :

*Proposition 1.1.*

- (1) For any non-negative initial condition, the solutions of system (2) stay positive at any time and are bounded when  $t \rightarrow +\infty$ .
- (2) The set

$$\Omega = \{(X_0, S_1, X_1) \in \mathbb{R}_+^3 : Z = k_0X_0 + S_1 + k_1X_1 \leq \max(Z(0), \frac{S_{1in}}{\alpha D})\}$$

is positively invariant and attractor of all solutions of (2), with  $S_{in} = D(k_0X_{0in} + S_{1in})$ .

## 2. STUDY OF THE SUB-MODEL

The washout equilibrium  $E_0 = (\frac{X_{0in}}{\alpha}, S_{1in}, 0)$ , always exists. To look for positive equilibria, we consider the function

$$\xi(X_0) = \frac{D(X_{0in} - \alpha X_0)}{\mu_0(X_0)}.$$

We assume that

**H0:** The function  $\mu_0(\cdot)$  is increasing,  $\mu_0(0) = 0$  and  $\mu_0''(X_0) \leq 0$  for all  $X_0 \in ]0, \frac{X_{0in}}{\alpha}]$ .

*Lemma 2.1.* Under assumption **H0**, the function  $\xi(\cdot)$  vanishes on  $\frac{X_{0in}}{\alpha}$ , is decreasing and convex.

In the case where the function  $\mu_0(\cdot)$  is linear or of Monod type, the assumption **H0** is satisfied.

### 2.1 Study of the sub-model with monotonic growth rate $\mu_1(\cdot)$

In this section, we study the existence of equilibrium points of system (2) under the following assumption

**H1:**  $\mu_1(0) = 0$  and  $\mu_1'(S_1) > 0$  for all  $S_1 \geq 0$ .

**H2:** The equation  $\mu_1(S_1) = \alpha D$  has a finite solution  $\lambda_1 = \mu_1^{-1}(\alpha D)$ .

Let  $\Delta$  the line of equation :

$$X_1 = \delta(X_0) = \frac{1}{k_1\alpha} [(S_{1in} - \lambda_1) + k_0(X_{0in} - \alpha X_0)].$$

*Lemma 2.2.* The equation  $\xi'(X_0) = -\frac{k_0}{k_1}$  has a unique solution  $\bar{X}_0 \in ]0, \frac{X_{0in}}{\alpha}]$  if and only if

$$\xi' \left( \frac{X_{0in}}{\alpha} \right) > -\frac{k_0}{k_1}.$$

Moreover,

$$\xi' \left( \frac{X_{0in}}{\alpha} \right) > -\frac{k_0}{k_1} \iff k_0\mu_0 \left( \frac{X_{0in}}{\alpha} \right) > k_1\alpha D.$$

If  $\xi' \left( \frac{X_{0in}}{\alpha} \right) > -\frac{k_0}{k_1}$ , the intersection of the line  $\Delta$  with the curve of the function  $\xi$ , has at most two points. Let us denote by  $E_1^* = (X_0^*, \lambda_1, X_1^*)$  and  $E_1^{**} = (X_0^{**}, \lambda_1, X_1^{**})$  (see Fig. 2). By Lemma 2.2, there exists a unique solution  $\bar{X}_0 \in ]0, \frac{X_{0in}}{\alpha}]$  of equation  $\xi'(X_0) = -\frac{k_0}{k_1}$ . Thus, there is a

limit value  $X_1^{Min}$  for which the curve  $\xi$  is tangent to the line  $\Delta$  and who satisfied

$$X_1^{Min} = \bar{X}_1 + \frac{k_0}{k_1} \bar{X}_0$$

with  $\bar{E}_1 = (\bar{X}_0, \lambda_1, \bar{X}_1)$  an equilibrium of (2) (see case 4 of the Fig. 3). At this limite value  $X_1^{Min}$ , we associate  $S_{1in}^{Min}$  which satisfied

$$X_1^{Min} = \delta(0) = \frac{1}{k_1 \alpha} [(S_{1in}^{Min} - \lambda_1) + k_0 X_{0in}].$$

In the generic case where  $S_{1in}^{Min} > 0$ , we have shown the following result :

*Proposition 2.1.*

- If  $\lambda_1 \leq S_{1in}$ , there exists a unique positive equilibrium  $E_1^* = (X_0^*, \lambda_1, X_1^*)$ .
- If  $S_1^{Min} < S_{1in} < \lambda_1$ , there exist two positive equilibria  $E_1^*$  et  $E_1^{**}$ .
- If  $S_{1in} = S_1^{Min}$ , there exists a unique positive equilibrium  $\bar{E}_1 = (\bar{X}_0, \lambda_1, \bar{X}_1)$ .
- If  $S_{1in} < S_1^{Min}$ , there is no positive equilibrium.

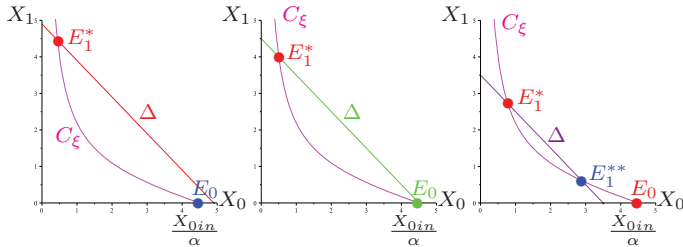


Fig. 2. Case 1:  $\lambda_1 < S_{1in} = 5.5$ , case 2:  $S_{1in} = 4.5 = \lambda_1$ , case 3:  $S_{1in} < S_{1in} = 2 < \lambda_1$ .

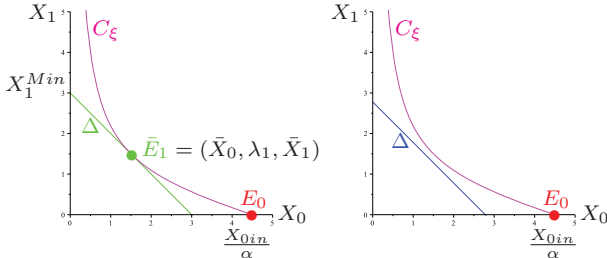


Fig. 3. Case 4:  $S_{1in} = 0.75 = S_{1in}^{Min}$ , case 5:  $S_{1in} = 0.2 < S_{1in}^{Min}$ .

In the following, we show the asymptotic behavior of equilibrium points. We choose the red color for Locally Asymptotically Stable (LAS) equilibrium, the green color for saddle node equilibrium and blue color for unstable equilibrium.

*Proposition 2.2.*

- $E_0$  is LAS if and only if  $\mu_1(S_{1in}) < \alpha D$ .
- If  $E_1^*$  exists, then it is LAS (stable node).
- If  $E_1^{**}$  exists, then it is unstable (saddle point).

**Proof.** At washout  $E_0 = (\frac{X_{0in}}{\alpha}, S_{1in}, 0)$ , the Jacobian matrix of (2) is

$$\mathbf{J}_0 = \begin{bmatrix} -\alpha D & 0 & -\mu_0(\frac{X_{0in}}{\alpha}) \\ 0 & -D & k_0 \mu_0(\frac{X_{0in}}{\alpha}) - k_1 \mu_1(S_{1in}) \\ 0 & 0 & \mu_1(S_{1in}) - \alpha D \end{bmatrix}$$

The eigenvalues are  $-\alpha D$ ,  $-D$  and  $\mu_1(S_{1in}) - \alpha D$ . Then, the equilibrium  $E_0$  is LAS if and only if  $\mu_1(S_{1in}) < \alpha D$ .

At a positive equilibrium  $E_1^* = (X_0^*, \lambda_1, X_1^*)$ , the Jacobian matrix is

$$\mathbf{J}_1 = \begin{bmatrix} -m_{11} & 0 & -m_{13} \\ m_{21} & -m_{22} & \theta \\ 0 & m_{32} & 0 \end{bmatrix}$$

where

$$\begin{aligned} m_{11} &= \alpha D + \mu'_0(X_0^*) X_1^*, & m_{13} &= \mu_0(X_0^*), \\ m_{21} &= k_0 \mu'_0(X_0^*) X_1^*, & m_{22} &= D + k_1 \mu'_1(\lambda_1) X_1^*, \\ \theta &= k_0 \mu_0(X_0^*) - k_1 \alpha D, & m_{32} &= \mu'_1(\lambda_1) X_1^*, \end{aligned}$$

with  $m_{11}, m_{13}, m_{21}, m_{22}$  and  $m_{32}$  are positive. The characteristic polynomial of  $J_1$  is given by

$$P_{J_1}(\lambda) = a_0 \lambda^3 + a_1 \lambda^2 + a_2 \lambda + a_3$$

with

$$\begin{aligned} a_0 &= -1, & a_1 &= -(m_{11} + m_{22}), & a_2 &= -m_{11} m_{22} + \theta m_{32}, \\ & & a_3 &= -m_{32} (m_{21} m_{13} - \theta m_{11}). \end{aligned}$$

According to the Routh-Hurwitz criterion,  $E_1^*$  is LAS if and only if

$$\begin{cases} a_i < 0, & i = 0 \dots 3 \\ a_1 a_2 - a_0 a_3 > 0. \end{cases}$$

We have

$$\begin{aligned} a_2 &= [k_0 \mu_0(X_0^*) - k_1 \alpha D - k_1 \mu'_0(X_0^*) X_1^*] \mu'_1(\lambda_1) X_1^* \\ &\quad - [D \mu'_0(X_0^*) X_1^* + \alpha D (D + k_1 \mu'_1(\lambda_1) X_1^*)]. \end{aligned}$$

Since

$$\xi'(X_0^*) + \frac{k_0}{k_1} = \frac{-k_1 \alpha D - k_1 \mu'_0(X_0^*) X_1^* + k_0 \mu_0(X_0^*)}{k_1 \mu_0(X_0^*)}$$

then we deduce

$$\begin{aligned} a_2 &= [\xi'(X_0^*) + \frac{k_0}{k_1}] k_1 \mu_0(X_0^*) \mu'_1(\lambda_1) X_1^* \\ &\quad - [D \mu'_0(X_0^*) X_1^* + \alpha D (D + k_1 \mu'_1(\lambda_1) X_1^*)] \end{aligned}$$

therefore, if  $\xi'(X_0^*) < -\frac{k_0}{k_1}$ , then  $a_2 < 0$ . Moreover

$$\begin{aligned} a_3 &= m_{32} \alpha D [k_0 \mu_0(X_0^*) - k_1 \alpha D - k_1 \mu'_0(X_0^*) X_1^*] \\ &= m_{32} \alpha D [\xi'(X_0^*) + \frac{k_0}{k_1}] k_1 \mu_0(X_0^*) \end{aligned}$$

is negative if and only if  $\xi'(X_0^*) < -\frac{k_0}{k_1}$ . Finally

$$\begin{aligned} a_1 a_2 - a_0 a_3 &= -[\xi'(X_0^*) + \frac{k_0}{k_1}] k_1 \mu_0(X_0^*) \mu'_0(X_0^*) \mu'_1(\lambda_1) X_1^{*2} \\ &\quad + P \end{aligned}$$

where

$$P = D m_{11}^2 + ((\alpha D)^2 + \alpha D \mu'_0(X_0^*) X_1^*) k_1 \mu'_1(\lambda_1) X_1^* - m_{22} a_2$$

is positive if  $\xi'(X_0^*) < -\frac{k_0}{k_1}$ . Since  $E_1^*$  satisfies  $\xi'(X_0^*) < -\frac{k_0}{k_1}$ , then it is LAS and  $E_1^{**}$  satisfies  $\xi'(X_0^{**}) > -\frac{k_0}{k_1}$ , then it is unstable.

The simulations shown in Fig. 4 where obtained for the following Monod functions

$$\mu_0(X_0) = \frac{2.5 X_0}{1.5 + X_0} \quad \text{and} \quad \mu_1(S) = \frac{2S}{1.5 + S}$$

and the following values of the parameters

$$X_{0in} = 3, \quad D = 1, \quad \alpha = 0.75, \quad k_0 = 1 \quad \text{and} \quad k_1 = 1.2$$

The value of the break-even concentration is  $\lambda_1 = 0.9$ . We illustrate the case of bistability for  $S_{1in} = 0.7$  such as

$$\xi'\left(\frac{X_{0in}}{\alpha}\right) = -0.412 > -0.833 = -\frac{k_0}{k_1}.$$

In this case, the equation  $\xi(X_0) = \delta(X_0)$  admits two solutions (see Fig. 4 on the left) and the system (2) has a washout equilibrium

$$E_0 = (4, 0.5, 0)$$

and two positive equilibria

$$E_1^* = (1.201, 0.9, 1.887), \quad E_1^{**} = (2.808, 0.9, 0.548).$$

The Fig. 4 in the middle shows the convergence to the positive equilibrium  $E_1^*$  for the initial condition

$$X_0(0) = 4.5, \quad S_1(0) = 2 \quad \text{and} \quad X_1(0) = 0.368$$

and the washout for the initial condition (see Fig. 4 on the right)

$$X_0(0) = 4.5, \quad S_1(0) = 2 \quad \text{and} \quad X_1(0) = 0.367$$

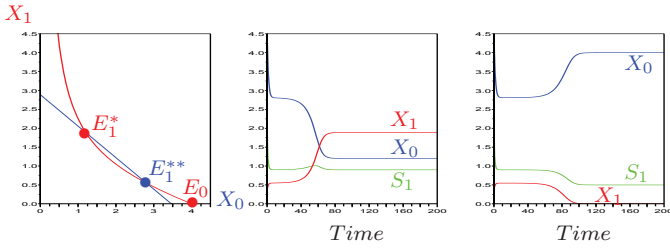


Fig. 4. Existence of two positive equilibria and bistability for  $S_{1in} < \lambda_1$ .

## 2.2 Study of the sub-model with non-monotonic growth rate $\mu_1(\cdot)$

In the following, we study the existence of equilibrium points of system (2) under the following assumption

**H3:** The function  $\mu_1(\cdot)$  is non monotonic and is such that the equation  $\mu_1(S_1) = \alpha D$  admits two solutions  $\lambda_1$  and  $\lambda_2$  with  $\lambda_1 < \lambda_2$ .

Let  $\Delta_i$  the line of equation :

$$X_1 = \delta_i(X_0) = \frac{1}{k_1 \alpha} \left[ (S_{1in} - \lambda_i) + k_0(X_{0in} - \alpha X_0) \right], \quad i = 1, 2.$$

The positive equilibrium  $E_i^* = (X_{0i}^*, \lambda_i, X_{1i}^*)$  of system (2) is a solution of the equation

$$X_{1i}^* = \delta_i(X_{0i}^*) = \xi(X_{0i}^*), \quad i = 1, 2.$$

When the intersection of the curve  $\xi$  and the line  $\Delta_i$  is formed by two points, let us denote them by  $E_i^* = (X_{0i}^*, \lambda_i, X_{1i}^*)$  if  $\xi'(X_{0i}^*) < -\frac{k_0}{k_1}$  and by  $E_i^{**} = (X_{0i}^{**}, \lambda_i, X_{1i}^{**})$ , otherwise (see Fig. 5 and Fig. 6). In the case  $\xi'(\frac{X_{0in}}{\alpha}) > -\frac{k_0}{k_1}$ , there exists a unique solution  $\bar{X}_0 \in ]0, \frac{X_{0in}}{\alpha}[$  of the equation  $\xi'(X_0) = -\frac{k_0}{k_1}$ . As the two lines  $\Delta_1$  and  $\Delta_2$  are parallel, then there exists for  $i = 1, 2$  a limit value  $X_1^{Min}$  for which the curve of  $\xi$  is tangent to the line  $\Delta_i$  and which satisfies

$$X_1^{Min} = \bar{X}_1 + \frac{k_0}{k_1} \bar{X}_0$$

with  $\bar{E}_i = (\bar{X}_0, \lambda_i, \bar{X}_1)$  an equilibrium of (2) for  $i = 1, 2$ . At this limit value  $X_1^{Min}$ , we associate  $S_{1i}^{Min}$  such that

$$X_1^{Min} = \delta(0) = \frac{1}{k_1 \alpha} [S_{1i}^{Min} - \lambda_i + k_0 X_{0in}].$$

In the generic case where  $S_{1i}^{Min} > 0$  for  $i = 1, 2$ , the cases  $S_{12}^{Min} < \lambda_1$  and  $S_{12}^{Min} > \lambda_1$  have to be distinguished. When  $S_{12}^{Min} < \lambda_1$ , we have shown the following result :

### Proposition 2.3.

- If  $S_{1in} \geq \lambda_2 > \lambda_1$ , there exist two positive equilibrium  $E_i^*$ ,  $i = 1, 2$ .
- If  $\lambda_1 \leq S_{1in} < \lambda_2$ , there exist three positive equilibrium  $E_i^*$  and  $E_2^{**}$ .
- If  $S_{12}^{Min} < S_{1in} < \lambda_1$ , there exist four positive equilibrium  $E_i^*$  and  $E_i^{**}$ .
- If  $S_{12}^{Min} = S_{1in}$ , there exist three positive equilibrium  $E_1^*$ ,  $E_1^{**}$  and  $E_2$ .
- If  $S_{11}^{Min} < S_{1in} < S_{12}^{Min}$ , there exist two positive equilibrium  $E_1^*$  and  $E_1^{**}$ .
- If  $S_{11}^{Min} = S_{1in}$ , there exists a unique positive equilibrium  $\bar{E}_1$ .
- If  $S_{1in} < S_{11}^{Min}$ , there is no positive equilibrium.

In the case where  $S_{12}^{Min} > \lambda_1$ , we can prove, similarly, that if  $S_{1in} \geq S_{11}^{Min}$ , we have one, two, three or four positive equilibria, depending on the position of  $S_{1in}$ .

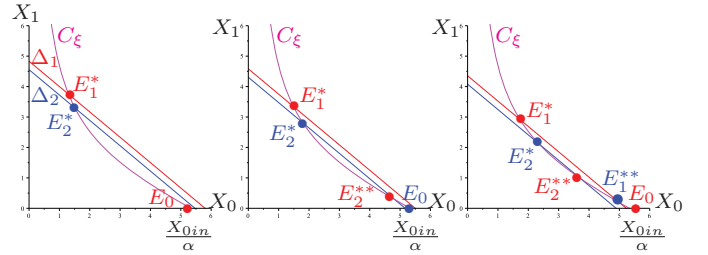


Fig. 5. Case 1:  $S_{1in} = 8 > \lambda_2 = 7$ , case 2:  $\lambda_1 = 4.5 < S_{1in} = 5.75 < \lambda_2$ , case 3:  $S_{12}^{Min} < S_{1in} = 3.7 < \lambda_1$ .

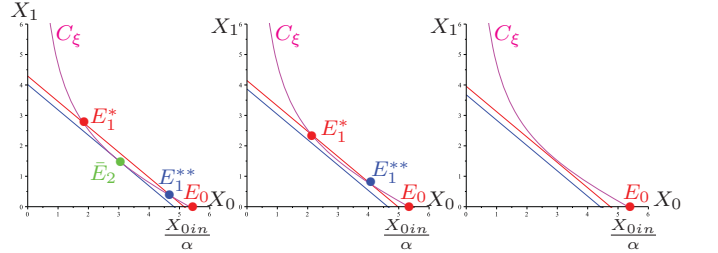


Fig. 6. Case 4:  $S_{1in} = 3.125 = S_{12}^{Min}$ , case 5:  $S_{11}^{Min} = 0.62 < S_{1in} = 1.8 < S_{12}^{Min}$ , case 6:  $S_{1in} = 0.1 < S_{11}^{Min}$ .

We study now the local stability properties of the equilibria. For  $E_0$ ,  $E_1^*$  and  $E_1^{**}$ , we have the same results as in Proposition 2.2. Moreover, we can prove :

### Proposition 2.4.

- If  $E_2^*$  exists, then it is unstable.
- If  $E_2^{**}$  exists and  $D + k_1 \mu_1'(\lambda_2) X_{12}^{**} > 0$ , then it is LAS.

**Proof.** The Jacobian of the system (2) at  $E_2^*$  is

$$\mathbf{J}_2 = \begin{bmatrix} -m_{11} & 0 & -m_{13} \\ m_{21} & \beta & \theta \\ 0 & -m_{32} & 0 \end{bmatrix}$$

where

$$\begin{aligned} m_{11} &= \alpha D + \mu_0'(X_{02}^*) X_{12}^*, & m_{13} &= \mu_0(X_{02}^*), \\ m_{21} &= k_0 \mu_0'(X_{02}^*) X_{12}^*, & \beta &= -(D + k_1 \mu_1'(\lambda_2) X_{12}^*), \\ \theta &= k_0 \mu_0(X_{02}^*) - k_1 \alpha D, & m_{32} &= -\mu_1'(\lambda_2) X_{12}^*, \end{aligned}$$

with  $m_{11}, m_{13}, m_{21}$  and  $m_{32}$  are positive. The characteristic polynomial of  $J_2$  is given by

$$P_{J_2}(\lambda) = a_0 \lambda^3 + a_1 \lambda^2 + a_2 \lambda + a_3$$



with

$$\begin{aligned} a_0 &= -1, & a_1 &= \beta - m_{11}, & a_2 &= m_{11}\beta - \theta m_{32}, \\ a_3 &= m_{32}(m_{21}m_{13} - \theta m_{11}). \end{aligned}$$

We have

$$\begin{aligned} a_2 &= \left[ \xi'(X_{02}^*) + \frac{k_0}{k_1} \right] k_1 \mu_0(X_{02}^*) \mu_1'(\lambda_2) X_{12}^* \\ &\quad - \left[ D \mu_0'(X_{02}^*) X_{12}^* + \alpha D (D + k_1 \mu_1'(\lambda_2) X_{12}^*) \right]. \end{aligned}$$

If  $D + k_1 \mu_1'(\lambda_2) X_{12}^* > 0$  and  $\xi'(X_{02}^*) > -\frac{k_0}{k_1}$ , then  $a_2 < 0$ . It is easy to check that

$$a_3 = -m_{32} \alpha D \left[ \xi'(X_{02}^*) + \frac{k_0}{k_1} \right] k_1 \mu_0(X_{02}^*)$$

who is negative if and only  $\xi'(X_{02}^*) > -\frac{k_0}{k_1}$ . Thus,  $E_2^*$  is unstable. One can readily check that

$$a_1 a_2 - a_0 a_3 = m_{11} \beta^2 - m_{11}^2 \beta + m_{32}(m_{21}m_{13} - \theta \beta)$$

and

$$\theta X_{12}^{**} = k_0 \mu_0(X_{02}^{**}) X_{12}^{**} - k_1 \mu_1(\lambda_2) X_{12}^{**} = -D(S_{1in} - \lambda_2).$$

If  $E_2^{**}$  exists then  $S_{1in} < \lambda_2$ , thus  $\theta > 0$ . We conclude that according to the Routh-Hurwitz criterion, if  $E_2^{**}$  exists and

$$D + k_1 \mu_1'(\lambda_2) X_{12}^{**} > 0,$$

then it is LAS.

### 3. STUDY OF CONTOIS MODEL

In this section we study the case where the growth rate  $\mu_0(\cdot)$  depends on  $X_0$  and also on  $X_1$ . Models with such growth rates may be of interest if we consider that we work in a non homogeneous environment Lobry and Harmand (2006). The Contois function is an exemple of these growth rates. We consider the model

$$\begin{cases} \dot{X}_0 = D(X_{0in} - \alpha X_0) - \mu_0(X_0, X_1) X_1, \\ \dot{S}_1 = D(S_{1in} - S_1) + k_0 \mu_0(X_0, X_1) X_1 - k_1 \mu_1(S_1) X_1, \\ \dot{X}_1 = [\mu_1(S_1) - \alpha D] X_1. \end{cases} \quad (3)$$

Let us denote by

$$f(X_0, X_1) = \mu_0(X_0, X_1) X_1 - D(X_{0in} - \alpha X_0).$$

We assume that

**H4:**  $\mu_0(0, X_1) = 0$  and  $\mu_0(0, X_1) > 0$  for all  $X_0 > 0$  and all  $X_1 \geq 0$ .

**H5:**  $\frac{\partial \mu_0}{\partial X_0} > 0$  and  $\frac{\partial \mu_0}{\partial X_1} < 0$  for all  $X_0 > 0$  and all  $X_1 \geq 0$ .

**H6:**  $\mu_1(0) = 0$ ,  $\mu_1'(S_1) > 0$  for all  $S_1 \geq 0$  and the equation  $\mu_1(S_1) = \alpha D$  has a unique solution  $\lambda_1$ .

**H7:**  $\frac{\partial f}{\partial X_1}(X_0, X_1) = \mu_0(X_0, X_1) + \frac{\partial \mu_0}{\partial X_1} X_1 > 0$  for all  $X_0 > 0$  and all  $X_1 \geq 0$ .

**H8:** There exist  $a \in ]0, \frac{X_{0in}}{\alpha}[$  such that  $\lim_{\substack{X_0 \rightarrow a \\ X_1 \rightarrow +\infty}} f(X_0, X_1) = 0$ .

We consider, now, the existence of positive equilibria. We first prove :

*Lemma 3.1.* Assume that **H4-H8** hold. The equation  $f(X_0, X_1) = 0$  defines a decreasing function

$$\begin{aligned} F : ]a, \frac{X_{0in}}{\alpha}] &\longrightarrow \mathbb{R}_+ \\ X_0 &\longrightarrow F(X_0) = X_1 \end{aligned}$$

where  $0 < a < \frac{X_{0in}}{\alpha}$  and such that

$$\lim_{X_0 \rightarrow a} F(X_0) = +\infty, \quad F\left(\frac{X_{0in}}{\alpha}\right) = 0 \quad \text{and} \quad F'(X_0) < 0.$$

Then we state the following result :

*Proposition 3.1.*

- For  $S_{1in} > \lambda_1$ , if  $F''(X_0) > 0$  for all  $X_0 \in ]a, \frac{X_{0in}}{\alpha}]$ , then there exists a unique positive equilibrium. If  $F''(X_0)$  changes sign for  $X_0 \in ]a, \frac{X_{0in}}{\alpha}]$ , then there exists at least one positive equilibrium. Generically one has an odd number of positive equilibria (see Fig. 7).
- For  $S_{1in} < \lambda_1$ , if  $F''(X_0) > 0$  for all  $X_0 \in ]a, \frac{X_{0in}}{\alpha}]$ , then there exist at most two positive equilibria. If  $F''(X_0)$  changes sign for  $X_0 \in ]a, \frac{X_{0in}}{\alpha}]$ , then the system has generically no positive equilibria or an even number of positive equilibria (see Fig. 8).

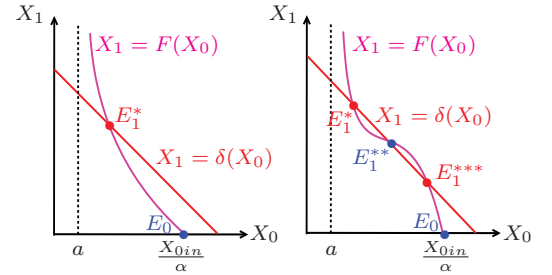


Fig. 7. Null-clines  $X_1 = F(X_0)$  and  $X_1 = \delta(X_0)$  for  $S_{1in} > \lambda_1$ . On the left,  $F(\cdot)$  concave. On the right,  $F''(\cdot)$  changes sign.

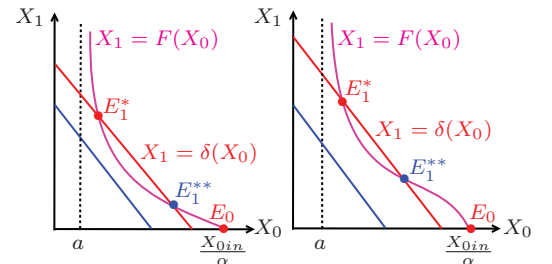


Fig. 8. Null-clines  $X_1 = F(X_0)$  and  $X_1 = \delta(X_0)$  for  $S_{1in} < \lambda_1$ . On the left,  $F(\cdot)$  concave. On the right,  $F''(\cdot)$  changes sign.

**Example** We consider the following function of Contois

$$\mu_0(X_0, X_1) = \frac{m_0 X_0}{X_0 + a_0 X_1}$$

where  $m_0 > 0$  and  $a_0 > 0$ . We have

$$\frac{\partial f}{\partial X_1}(X_0, X_1) = \frac{m_0 X_0^2}{(X_0 + a_0 X_1)^2} > 0$$

thus Assumption **H7** is satisfied. Moreover,

$$\lim_{X_1 \rightarrow +\infty} f(X_0, X_1) = 0$$

$$\implies X_0 = \frac{DX_{0in}}{\frac{m_0}{a_0} + \alpha D} = a$$

where  $0 < a < \frac{X_{0in}}{\alpha}$ . Therefore, there exists  $a \in ]0, \frac{X_{0in}}{\alpha}[$  such that  $\lim_{\substack{X_0 \rightarrow a \\ X_1 \rightarrow +\infty}} \tilde{f}(X_0, X_1) = 0$  i.e.,  $\lim_{X_0 \rightarrow a} F(X_0) = +\infty$ .

By the implicit function theorem, we have

$$F'(X_0) = -\frac{m_0 a_0 X_1^2 + \alpha D (X_0 + a_0 X_1)^2}{m_0 X_0^2} < 0$$

and

$$F''(X_0) = \frac{2a_0 a_0 X_1 [m_0 X_1 + \alpha D (X_0 + a_0 X_1)]}{m_0 X_0^3} > 0.$$

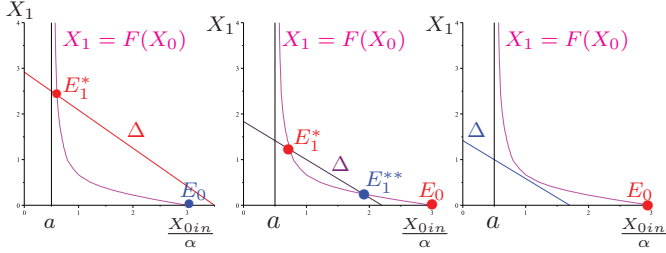


Fig. 9. Cas 1:  $S_{1in} = 2 > \lambda_1 = 1.5$ , cas 2:  $S_{1in} = 0.7 < \lambda_1$ , cas 3:  $S_{1in} = 0.2$ .

The stability results we obtain in this case are given by :

*Proposition 3.2.*

- The washout equilibrium  $E_0$  is LAS if and only if  $\mu_1(S_{1in}) < \alpha D$ .
- If the positive equilibrium  $E_1^*$  exists, then it is LAS (stable node).
- If the positive equilibrium  $E_1^{**}$  exists, then it is unstable (saddle point).

**Proof.** At washout  $E_0 = (\frac{X_{0in}}{\alpha}, S_{1in}, 0)$ , the Jacobian matrix is

$$\mathbf{J}_0 = \begin{bmatrix} -\alpha D & 0 & -\mu_0(\frac{X_{0in}}{\alpha}, 0) \\ 0 & -D & k_0 \mu_0(\frac{X_{0in}}{\alpha}, 0) - k_1 \mu_1(S_{1in}) \\ 0 & 0 & \mu_1(S_{1in}) - \alpha D \end{bmatrix}$$

The eigenvalues are  $-\alpha D$ ,  $-D$  and  $\mu_1(S_{1in}) - \alpha D$ . Then, the equilibrium  $E_0$  is LAS if and only if  $\mu_1(S_{1in}) < \alpha D$ .

At a positive equilibrium  $E_1^* = (X_0^*, \lambda_1, X_1^*)$ , the Jacobian matrix is

$$\mathbf{J}_1 = \begin{bmatrix} -m_{11} & 0 & -m_{13} \\ m_{21} & -m_{22} & \theta \\ 0 & m_{32} & 0 \end{bmatrix}$$

where

$$m_{11} = \alpha D + \frac{\partial \mu_0}{\partial X_0} X_1^*, \quad m_{13} = \mu_0(X_0^*, X_1^*) + \frac{\partial \mu_0}{\partial X_1} X_1^*,$$

$$m_{21} = k_0 \frac{\partial \mu_0}{\partial X_0} X_1^*, \quad m_{22} = D + k_1 \mu_1'(\lambda_1) X_1^*,$$

$$\theta = k_0 \frac{\partial \mu_0}{\partial X_1} X_1^* + k_0 \mu_0(X_0^*, X_1^*) - k_1 \alpha D, \quad m_{32} = \mu_1'(\lambda_1) X_1^*,$$

with  $m_{11}, m_{13}, m_{21}, m_{22}$  and  $m_{32}$  are positive. The characteristic polynomial of  $\mathbf{J}_1$  is given by

$$P_{J_1}(\lambda) = a_0 \lambda^3 + a_1 \lambda^2 + a_2 \lambda + a_3$$

with

$$a_0 = -1, \quad a_1 = -(m_{11} + m_{22}), \quad a_2 = -m_{11} m_{22} + \theta m_{32}, \\ a_3 = -m_{32} (m_{21} m_{13} - \theta m_{11}).$$

One can easily check that

$$a_2 = \left[ F'(X_0^*) + \frac{k_0}{k_1} \right] k_1 \frac{\partial f}{\partial X_1} \mu_1'(\lambda_1) X_1^* \\ - \left( D \frac{\partial \mu_0}{\partial X_0} X_1^* + \alpha D m_{22} \right).$$

Therefore, if  $F'(X_0^*) < -\frac{k_0}{k_1}$ , then  $a_2 < 0$ . One can also check that

$$a_3 = m_{32} \alpha D \left[ F'(X_0^*) + \frac{k_0}{k_1} \right] k_1 \frac{\partial f}{\partial X_1}$$

is negative if and only if  $F'(X_0^*) < -\frac{k_0}{k_1}$ . So,  $E_1^{**}$  is unstable. Finally

$$a_1 a_2 - a_0 a_3 = - \left[ F'(X_0^*) + \frac{k_0}{k_1} \right] k_1 \frac{\partial f}{\partial X_1} \frac{\partial \mu_0}{\partial X_0} \mu_1'(\lambda_1) X_1^{*2} \\ + P$$

where

$$P = D m_{11}^2 + \left[ (\alpha D)^2 + \alpha D \frac{\partial \mu_0}{\partial X_0} X_1^* \right] k_1 \mu_1'(\lambda_1) X_1^* - m_{22} a_2$$

is positive if  $F'(X_0^*) < -\frac{k_0}{k_1}$ . Since  $E_1^*$  satisfy  $\xi'(X_0^*) < -\frac{k_0}{k_1}$ , then according to the Routh-Hurwitz criterion, it is LAS.

## 4. CONCLUSION

In this work, we have analyzed a model of the chemostat with enzymatic degradation of a substrate that can partly be under a solid form. The sub-model studied with monotonic growth rates may exhibit a bi-stable behavior, while it may only occur in the classical chemostat model when the growth rate is non monotonic. We also studied the case where the growth rate is density-dependent. Depending on the input substrate concentration, it was shown that the system can exhibit either a bistability or the global stability of the positive equilibrium point or of the washout.

## REFERENCES

- B. Benyahia, T. Sari, B.C. and Harmand, J. (2010a). Equilibria of an anaerobic wastewater treatment process and their stability. In J.V.I.D.D. JR. Banga P. Bogaerts and I. Smets (eds.), *11th International Symposium on Computer Applications in Biotechnology*, 371–376. International Federation of Automatic Control.
- B. Benyahia, T. Sari, B.C. and Harmand, J. (2010b). Sur le modèle am2 de digestion anaérobie. In A.S. E. Badouel and I. Lokpo (eds.), *CARI'10, Proceedings of the 10th African Conference on Research in Computer Science and Applied Mathematics*, 453–460. INRIA.
- Bastin, G. and Dochain, D. (1991). On-line estimation and adaptive control of bioreactors, dynamics of microbial competition. *Elsevier Science Publishers, A. and N.Y.*
- Lobry, C. and Harmand, J. (2006). A new hypothesis to explain the coexistence of n species in the presence of a single resource. *CRAS série Biologie*.
- Simeonov, I. and Stoyanov, S. (2003). Modelling and dynamic compensator control of the anaerobic digestion of organic wastes. *Chem. Biochem. Eng. Q.*
- Smith, H. and Waltman, P. (1995). *The Theory of the Chemostat, Dynamics of Microbial Competition*. Cambridge University Press.